Supported by tetfund TERTIARY EDUCATION TRUST FUND

ANALYSES OF STATES, CONTROLS AND COST FUNCTIONAL IN ONE DIMENSIONAL WAVE EQUATION WITH ENERGY EFFECT UP TO TEN NODAL POINTS

Musa Bawa

Department of Mathematics/Computer Science, Ibrahim Badamasi Babangida University, Lapai, Nigeria

musa_bawa@yahoo.com

Received: August 25, 2016 **Accepted:** February 19, 2017

Abstract: The fourier solution proposed by Duchateau and Zachmann for deriving the general equations for states and controls was applied to the problems of one dimensional quadratic functionalMin $\int$ [z,u] =Min $\int_0^1\int_0^1 \{u^2(x,t)+z^2(x,t)\}$dxdt and the finite element technique used on the resulting system to obtain the states, controls and the cost functional at different levels of discretization up to ten nodal points. The numerical solutions depict increase in the cost functional as the space dimension increases while as the number of elements increase, the controls get smaller among other things.

Keywords: Cost functional, nodal points, optimal control, optimal state, wave equation

Introduction

Solutions of wave equations with energy effects have been considered by many authors over the period, such as Binder (1911), Reju (1995), Schmidt (1924), and Bawa (2013). In this work, the procedural steps applied is in a manner similar to that of Bawa (2013) in order to obtain the extremals-control(U_{n_e}), state (Z_{n_e})and the cost functional (J_{n_e}) at different level profile up to ten nodal points. In theoretical physics and engineering, partial differential equations generally arise from the mathematical formulation of real life physical problems as in Raisinghania (2010).

Statement of the Problem

Cases of manageable energy losses can be modeled as a combination of the wave and energy equations simply; this is considered as wave equation with energy effect or energized wave equation. According to pain (1976), the one-dimensional wave equation with energy effect is given by:

$$\frac{\partial^2 z(x,t)}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 z(x,t)}{\partial t^2} + \frac{1}{d}\frac{\partial z(x,t)}{\partial t} \qquad (1)$$

Where: C and d are the material property Constants namely the wave velocity and energy, respectively.

The wave part is:

$$\frac{\partial^2 z(x,t)}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 z(x,t)}{\partial t^2} \qquad (2)$$

And the energy part is:

$$\frac{\partial^2 z(x,t)}{\partial x^2} = \frac{1}{d}\frac{\partial z(x,t)}{\partial t} \qquad (3)$$

The optimization problem under consideration is:

$$\text{Min}\int [z,u] = \text{Min}\int_0^1\int_0^1 \{u^2(x,t)+z^2(x,t)\}\text{dxdt} \qquad (4)$$

Subject to:

$$\frac{\partial^2 z(x,t)}{\partial x^2} + \frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + u(x,t) \qquad (5)$$

With boundary and initial conditions:

$$Z(0,t) = z(1,t) = 0\ ;\ 0\le t\le 1$$
$$Z(x,0) = Z_o(x);\quad 0\le x\le 1$$

Where: u(x,t) is the Control or input function.

Writing the Hamiltonian for (4) and (5), similar to that of Singh and Titli (1978), we have:

$$H = Z^2(x,t) + U^2(x,t) + \lambda^T\left[\frac{\partial^2 z(x,t)}{\partial x^2} + u(x,t)\right] \qquad (6)$$

Where: $\lambda^T = \lambda^T(t)$

Setting:

$$F[z(x,t),u(x,t)] = \frac{\partial^2 z(x,t)}{\partial x^2} + u(x,t)$$

And:

g[(z(x,t),u(x,t)] = z^2 (x,t) + u^2(x,t), consequently, we have the first order necessary conditions for optimization as:

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial H(x,t)}{\partial \lambda} = \frac{\partial^2 z(x,t)}{\partial x^2} + u(x,t) = F[z(x,t),u(x,t)] \quad (7)$$

$$\frac{\partial \lambda}{\partial t} = -\frac{\partial H}{\partial Z} = -\left[\frac{\partial f}{\partial z}\right]^T - \frac{\partial g}{\partial z} = -2z(x,t) \qquad (8)$$

$$\frac{\partial H}{\partial u} = 0$$

$$\text{Or}\quad \left[\frac{\partial f}{\partial u}\right] + \frac{\partial g}{\partial u} = 0 \qquad (9)$$

Where: H = g(z,u) + λ^T(t) f (z,u)

Equation (9) gives:

$$\lambda + 2u(x,t) = 0 \qquad (10)$$

or $\lambda = -2u(x,t)$

Using equations (8) and (2.10), gives:

$$\frac{\partial \lambda}{\partial t} = \frac{2\partial u(t,x)}{\partial t} = -2z(x,t),$$

Which implies that:

$$Z(x,t) = -\frac{\partial u(t,x)}{\partial t} \qquad (11)$$

Equation (11) is here of physical significance under the conditions for optimality and it expresses the relationship between the temperature and the heat source at any point x of the unit propagating rod of our diffused model. Moreover, (11) is in this case treated as a differential transform of any previous known solution of the wave diffusion equation.

Assuming that (11) admits the Fourier solutions proposed by Duchateau and Zachmann (1986).

$$Z(x,t) = \sum_{i=1}^{\infty} \propto_1 (t)\sin\pi ix$$
$$u(x,t) = \sum_{i=1}^{\infty} U_i(t)\sin\pi ix \qquad (12)$$

Then, the new solution obtained is as:

$$Z(x,t) = \frac{\partial}{\partial t}\left[\sum_{i=1}^{\infty} U_i(t)\sin\pi ix\right]$$
$$= \sum_{i=1}^{\infty} U_{it}(t)\sin\pi ix \qquad (13)$$

It follows that:

$$\propto_1 (t) = U_{it} \ (t)$$

And:

$$Z_t \ (x,t) = \sum_{i=1}^{\infty} U_{itt} \ (t) sin\pi ix$$

$$Z_{tt} \ (x,t) = \sum_{i=1}^{\infty} U_{ittt} \ (t) sin\pi ix \qquad (14)$$

$$Z_{xx} \ (x,t) = \sum_{i=1}^{\infty} i^2(-\pi^2) U_{it}(t) sin\pi ix$$

$$Z \ \ (x,0) = \sum_{i=1}^{\infty} U_{it} \ (0) sin\pi ix$$

From our constrained equations:

$$Z_{tt}(x,t) + Z_t \ (x,t) = Z_{xx} \ (x,t) + U(x,t)$$

This implies that:

$$\sum_{i=1}^{\infty} U_{ttt} \ (t) sin\pi ix \ + \sum_{i=1}^{\infty} U_{tt} \ (t) sin\pi ix$$

$$= \sum_{i=1}^{\infty} i^2(-\pi^2) U_{tt}(t) sin\pi ix + \sum_{i=1}^{\infty} U_i \ (t) sin\pi ix$$

Thus, we have:

$$U_{ittt} \ = \ U_{itt} \ - i^2 \pi^2 U_{it} + \ U_i$$

Now, the problem can be put in the form:

$$Min \int_0^1 [U_1 + \ U_2 + \ ... + U_n] dt + \int_0^1 [U_{1t} + \ U_{2t} + \ ... + U_{nt}] dt +$$

$$\int_0^1 [U_{1tt} + \ U_{2tt} ... + U_{ntt}] dt \qquad (15)$$

Subject to the set of equations below:

$$U_{1ttt} = -U_{1tt} - \pi^2 1^2 U_{1t} + U_1$$

$$U_{2ttt} = -U_{2tt} - \pi^2 2^2 U_{2t} + U_2$$

$$\vdots \qquad \qquad \vdots$$

$$U_{nttt} = -U_{ntt} - \pi^2 n^2 U_{nt} + U_n \qquad (16)$$

Computational Results

System (16) is now solved applying the finite element technique as in Rao (1989) and Bawa (2013). The elements characteristic matrices and vectors are obtained and the overall equations given as

$$[K] U_n^{\rightarrow} = P^{\rightarrow} \qquad (17)$$

Where: K is the characteristic matrix and $P^{\rightarrow}$ is the characteristic vector. This is solved and the values of the states, controls and cost functional obtained up to ten nodal points as presented on the table of results below

	n = 1			n = 2		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$0.2727273\pi^2$	$-0.1363636\pi^2$	$0.0929752\pi^4$	$1.0909092\pi^2$	$-0.5454544\pi^2$	$1.487603385\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$0.367347\pi^2$	$-0.122449\pi^2$	$0.149937576\pi^4$	$1.469388\pi^2$	$-0.489796\pi^2$	$2.399001216\pi^4$
	0	$-0.122449\pi^2$	$0.014993757\pi^4$	0	$-0.489796\pi^2$	$0.239900121\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$0.4138932\pi^2$	$-0.1034733\pi^2$	$0.182014304\pi^4$	$1.4138932\pi^2$	$-0.4138932\pi^2$	$0.182014304\pi^4$
	$0.1408688\pi^2$	$-0.1386905\pi^2$	$0.039079073\pi^4$	$1.6555728\pi^2$	$-0.554762\pi^2$	$0.039079073\pi^4$
	$-0.1408688\pi^2$	$-0.1034733\pi^2$	$0.030550742\pi^4$	$-1.6555728\pi^2$	$-0.413893\pi^2$	$0.030550742\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$0.4413738\pi^2$	$-8.857476E-02\pi^2$	$0.283385600\pi^4$	$1.7654952\pi^2$	$-8.3542990\pi^2$	$3.242501083\pi^4$
	$0.2251602\pi^2$	$-0.1333068\pi^2$	$0.068467820\pi^4$	$0.9006408\pi^2$	$-0.5332272\pi^2$	$1.095485097\pi^4$
	0	$-0.1333068\pi^2$	$0.017770702\pi^4$	0	$-0.5332272\pi^2$	$0.284331246\pi^4$
	$-0.2251602\pi^2$	$-8.857476E-02\pi^2$	$0.058542600\pi^4$	$-0.9006408\pi^2$	$-0.3542990\pi^2$	$0.936681632\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$0.4594609\pi^2$	$-7.657682E-02\pi^2$	$0.216968328\pi^4$	$1.8378436\pi^2$	$-0.01914421\pi^2$	$3.378035599\pi^4$
	$0.2808586\pi^2$	$-0.1233866\pi^2$	$0.094105806\pi^4$	$1.1234344\pi^2$	$-0.493546\pi^2$	$1.505692900\pi^4$
	$0.0944904\pi^2$	$-0.139135\pi^2$	$0.028286983\pi^4$	$0.3779616\pi^2$	$-0.556540\pi^2$	$0.452591742\pi^4$
	$-0.0944904\pi^2$	$-0.1233866\pi^2$	$0.024152688\pi^4$	$-0.3779616\pi^2$	$-0.4935464\pi^2$	$0.386443020\pi^4$
	$-0.2808586\pi^2$	$-7.657682E-02\pi^2$	$0.084745500\pi^4$	$-1.1234344\pi^2$	$-0.01914421\pi^2$	$1.262471352\pi^4$
	0	0	0	0	0	0

n = 1	n = 2

	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0	0	0	0	0	0
	$0.4722524\pi^2$	$-6.746464E-02\pi^2$	$0.227573807\pi^4$	$1.8890096\pi^2$	$-0.26985856\pi^2$	$3.641180911\pi^4$
	$0.3202742\pi^2$	$-0.1132181\pi^2$	$0.115393901\pi^4$	$1.2810968\pi^2$	$-0.4528724\pi^2$	$1.846302422\pi^4$
	$0.1617826\pi^2$	$-0.1363299\pi^2$	$0.044759451\pi^4$	$0.6471364\pi^2$	$-0.5453196\pi^2$	$0.716151220\pi^4$
	0	$-0.1363299\pi^2$	$0.018584100\pi^4$	0	$-0.5453196\pi^2$	$0.297373466\pi^4$
	$-0.1617826\pi^2$	$-0.1132181\pi^2$	$0.038991947\pi^4$	$-0.6471304\pi^2$	$-0.4528724\pi^2$	$0.623871165\pi^4$
	$-0.3202742\pi^2$	$-6.746464E-02\pi^2$	$0.107127040\pi^4$	$-1.2810968\pi^2$	$-0.26985856\pi^2$	$1.714032653\pi^4$
	0	0	0	0	0	0
8	0	0	0	0	0	0
	$0.4817704\pi^2$	$-0.06022413\pi^2$	$0.235729664\pi^4$	$1.9270816\pi^2$	$-0.24089652\pi^2$	$3.771674626\pi^4$
	$0.3495872\pi^2$	$-0.1039197\pi^2$	$0.133010514\pi^4$	$1.3983488\pi^2$	$-0.4156788\pi^2$	$2.128168231\pi^4$
	$0.2119552\pi^2$	$-0.1304141\pi^2$	$0.065210918\pi^4$	$0.8478208\pi^2$	$-0.5216564\pi^2$	$0.990925508\pi^4$
	$0.0710208\pi^2$	$-0.1392917\pi^2$	$0.024446131\pi^4$	$0.2840832\pi^2$	$-0.5571668\pi^2$	$0.407521285\pi^4$
	$-0.0710208\pi^2$	$-0.1304141\pi^2$	$0.022051791\pi^4$	$-0.2840832\pi^2$	$-0.5216564\pi^2$	$0.352828664\pi^4$
	$-0.2119552\pi^2$	$-0.1039197\pi^2$	$0.055724310\pi^4$	$-0.8478208\pi^2$	$-0.4156788\pi^2$	$0.891588973\pi^4$
	$0.3495872\pi^2$	$-0.0602213\pi^2$	$0.125837815\pi^4$	$1.3983488\pi^2$	$-0.2408965\pi^2$	$2.013410490\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$0.4891257\pi^2$	$-5.434729E-02\pi^2$	$0.242197578\pi^4$	$1.9565028\pi^2$	$-0.21738916\pi^2$	$3.875161227\pi^4$
	$0.3722166\pi^2$	$-9.570468E-02\pi^2$	$0.147704583\pi^4$	$1.4888664\pi^2$	$-0.38281872\pi^2$	$2.363266183\pi^4$
	$0.2507221\pi^2$	$-0.1235627\pi^2$	$0.078129312\pi^4$	$1.002888\pi^2$	$-0.4942508\pi^2$	$1.250068194\pi^4$
	$0.1261377\pi^2$	$-0.137578\pi^2$	$0.034838425\pi^4$	$0.5045508\pi^2$	$-0.550312\pi^2$	$0.557414807\pi^4$
	0	$-0.137578\pi^2$	$0.018927706\pi^4$	0	$-0.550312\pi^2$	$0.302843297\pi^4$
	$-0.1261377\pi^2$	$-0.1235627\pi^2$	$0.031178460\pi^4$	$-0.5045508\pi^2$	$-0.4942508\pi^2$	$0.498855363\pi^4$
	$-0.2507221\pi^2$	$-9.570468E-02\pi^2$	$0.072020957\pi^4$	$-1.002888\pi^2$	$0.38281872\pi^2$	$1.691678379\pi^4$
	$-0.3722166\pi^2$	$-5.434729E-02\pi^2$	$0.141498378\pi^4$	$-1.4888664\pi^2$	$-0.21738916\pi^2$	$2.263974057\pi^4$
	0	0	0	0	0	0
10	0	0	0	0	0	0
	$0.4949809\pi^2$	$-4.949809E-02\pi^2$	$0.247456100\pi^4$	$1.9799236\pi^2$	$-0.19799236\pi^2$	$3.959307940\pi^4$
	$0.3902051\pi^2$	$-8.851864E-02\pi^2$	$0.160095500\pi^4$	$1.5608204\pi^2$	$-0.35407456\pi^2$	$2.561529115\pi^4$
	$0.2815346\pi^2$	$-0.1166721\pi^2$	$0.092874100\pi^4$	$1.1261384\pi^2$	$-0.4666884\pi^2$	$1.485981652\pi^4$
	$0.170053\pi^2$	$-0.1336774\pi^2$	$0.467876000\pi^4$	$0.680212\pi^2$	$-0.5347096\pi^2$	$0.748602721\pi^4$
	$0.056874\pi^2$	$-0.1393648\pi^2$	$0.022657100\pi^4$	$0.227496\pi^2$	$-0.5574592\pi^2$	$0.362515189\pi^4$
	$-0.056874\pi^2$	$-0.1336774\pi^2$	$0.021104300\pi^4$	$-0.227496\pi^2$	$-0.5347096\pi^2$	$0.337666056\pi^4$
	$-0.170053\pi^2$	$-0.1166721\pi^2$	$0.042530300\pi^4$	$-0.680212\pi^2$	$-0.4666884\pi^2$	$0.682499310\pi^4$
	$-0.2815346\pi^2$	$-8.851864E-02\pi^2$	$0.087097200\pi^4$	$-1.1261384\pi^2$	$-0.35407456\pi^2$	$1.3935556490\pi^4$
	$0.390205\pi^2$	$-4.949809E-02\pi^2$	$0.154710100\pi^4$	$1.5608204\pi^2$	$-0.19799236\pi^2$	$2.475334072\pi^4$
	0	0	0	0	0	0

	n = 3			n = 4		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$2.4545451\pi^2$	$-1.2272724\pi^2$	$7.53098870\pi^4$	$4.3636368\pi^2$	$-2.1818176\pi^2$	$23.801654160\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$3.306123\pi^2$	$-1.102041\pi^2$	$12.144943660\pi^4$	$5.877552\pi^2$	$-1.959184\pi^2$	$38.38401946\pi^4$
	0	$-1.102041\pi^2$	$12.144943660\pi^4$	0	$-1.959184\pi^2$	$3.838401946\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$3.7250388\pi^2$	$-0.9312597\pi^2$	$1.392925890\pi^4$	$6.6222912\pi^2$	$-1.6555728\pi^2$	$46.595662030\pi^4$
	$1.2678192\pi^2$	$-1.2482145\pi^2$	$3.165404962\pi^4$	$2.2539008\pi^2$	$-2.219048\pi^2$	$10.004242840\pi^4$
	$-1.2678192\pi^2$	$-0.9312597\pi^2$	$2.474610153\pi^4$	$-2.2539008\pi^2$	$-1.6555728\pi^2$	$10.004242840\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$3.9723642\pi^2$	$-0.7971728\pi^2$	$16.415162000\pi^4$	$7.067808\pi^2$	$-1.41719616\pi^2$	$51.0962354880\pi^4$
	$2.0264418\pi^2$	$-1.1997612\pi^2$	$5.545893300\pi^4$	$3.6025632\pi^2$	$-2.1329088\pi^2$	$17.527761560\pi^4$
	0	$-1.1997612\pi^2$	$1.439426900\pi^4$	0	$-2.1329088\pi^2$	$4.549299949\pi^4$
	$-2.0264418\pi^2$	$-0.7971728\pi^2$	$4.741950800\pi^4$	$-3.6025632\pi^2$	$-1.41719616\pi^2$	$17.527761560\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$4.1351481\pi^2$	$-0.6891913\pi^2$	$17.574434460\pi^4$	$7.3513744\pi^2$	$-1.2252288\pi^2$	$55.543891180\pi^4$
	$2.5277274\pi^2$	$-1.1104794\pi^2$	$7.622570307\pi^4$	$4.4937376\pi^2$	$-1.9741856\pi^2$	$24.091086400\pi^4$
	$0.8504136\pi^2$	$-1.252215\pi^2$	$2.291245697\pi^4$	$1.5118464\pi^2$	$-2.22616\pi^2$	$7.241467883\pi^4$
	$-0.8504136\pi^2$	$-1.1104794\pi^2$	$1.956367789\pi^4$	$-1.5118464\pi^2$	$-1.9741856\pi^2$	$6.203141703\pi^4$
	$-2.5277274\pi^2$	$-6.6891913\pi^2$	$51.134686060\pi^4$	$-4.4937376\pi^2$	$-1.225288\pi^2$	$21.694863230\pi^4$
	0	0	0	0	0	0
7	0	0	0	0	0	0
	$4.2502716\pi^2$	$-0.6071817\pi^2$	$18.433478290\pi^4$	$7.636032\pi^2$	$-1.07943424\pi^2$	$59.474162980\pi^4$
	$2.8824678\pi^2$	$-1.0189629\pi^2$	$9.327583518\pi^4$	$5.1243872\pi^2$	$-1.8114896\pi^2$	$29.540838750\pi^4$
	$1.4560434\pi^2$	$-1.22696911\pi^2$	$3.732177618\pi^4$	$2.5885216\pi^2$	$-2.1812784\pi^2$	$11.441317220\pi^4$
	0	$-1.22696911\pi^2$	$1.505447798\pi^4$	0	$-2.1812784\pi^2$	$4.757975458\pi^4$
	$-1.4560434\pi^2$	$1.0189629\pi^2$	$3.158347774\pi^4$	$-2.5885216\pi^2$	$-1.8114896\pi^2$	$9.9819364710\pi^4$
	$-2.8824678\pi^2$	$-0.6071817\pi^2$	$8.677290235\pi^4$	$-5.1243872\pi^2$	$-1.07943424\pi^2$	$27.424522370\pi^4$
	0	0	0	0	0	0

	n = 3			n = 4		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
8	0	0	0	0	0	0
	$4.3359336\pi^2$	$-0.5419917\pi^2$	$19.094075190\pi^4$	$7.7083264\pi^2$	$-0.9635408\pi^2$	$60.346706760\pi^4$
	$3.1462848\pi^2$	$-0.9352773\pi^2$	$10.773851670\pi^4$	$5.5933952\pi^2$	$-1.6627152\pi^2$	$34.050691700\pi^4$
	$1.9075968\pi^2$	$-1.1737269\pi^2$	$5.016560387\pi^4$	$3.3912832\pi^2$	$-2.0866256\pi^2$	$11.508305740\pi^4$
	$0.6391872\pi^2$	$-1.2536253\pi^2$	$1.980136669\pi^4$	$1.1363328\pi^2$	$-1.6627152\pi^2$	$4.055874069\pi^4$
	$-0.6391872\pi^2$	$-1.1737269\pi^2$	$1.786195112\pi^4$	$-1.1363328\pi^2$	$-2.0866256\pi^2$	$5.645258620\pi^4$
	$-1.9075968\pi^2$	$-0.9352773\pi^2$	$4.513669179\pi^4$	$-3.3912832\pi^2$	$-1.6627152\pi^2$	$14.265423580\pi^4$
	$-3.1462848\pi^2$	$-0.5419917\pi^2$	$10.192863050\pi^4$	$-5.5933952\pi^2$	$-0.9635408\pi^2$	$32.214480740\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$4.4021313\pi^2$	$-0.4891256\pi^2$	$19.618003840\pi^4$	$7.8260112\pi^2$	$-0.86955664\pi^2$	$62.002580050\pi^4$
	$3.3499494\pi^2$	$-0.8613421\pi^2$	$0.864374795\pi^4$	$5.9554656\pi^2$	$-1.53127488\pi^2$	$37.812373270\pi^4$
	$2.2564989\pi^2$	$-1.1120643\pi^2$	$6.328470231\pi^4$	$4.011552\pi^2$	$-1.9770032\pi^2$	$20.001091100\pi^4$
	$1.1352393\pi^2$	$-1.238202\pi^2$	$2.821912460\pi^4$	$2.0182032\pi^2$	$-2.201248\pi^2$	$8.918636914\pi^4$
	0	$-1.238202\pi^2$	$1.551441930\pi^4$	0	$-2.201248\pi^2$	$4.845492758\pi^4$
	$-1.1352393\pi^2$	$-1.1120643\pi^2$	$2.525455276\pi^4$	$-2.0182032\pi^2$	$-1.977003\pi^2$	$7.981685018\pi^4$
	$-2.2564989\pi^2$	$-0.8613421\pi^2$	$5.833697499\pi^4$	$-4.011552\pi^2$	$-153127488\pi^2$	$18.437352210\pi^4$
	$-3.3499494\pi^2$	$-0.4891256\pi^2$	$11.461404840\pi^4$	$-5.9554656\pi^2$	$-0.86955664\pi^2$	$36.223472480\pi^4$
	0	0	0	0	0	0
10	0	0	0	0	0	0
	$4.4548281\pi^2$	$-0.4454828\pi^2$	$20.043948000\pi^4$	$7.91968\pi^2$	$-0.79196944\pi^2$	$63.348546900\pi^4$
	$3.5118459\pi^2$	$-0.7966677\pi^2$	$12.967741000\pi^4$	$6.24328\pi^2$	$-1.41629824\pi^2$	$40.984445860\pi^4$
	$2.5338114\pi^2$	$-1.0500489\pi^2$	$7.522802900\pi^4$	$4.5045536\pi^2$	$-1.867536\pi^2$	$23.778693850\pi^4$
	$1.530477\pi^2$	$-1.2030966\pi^2$	$3.789801300\pi^4$	$2.720848\pi^2$	$-2.138832\pi^2$	$11.977616100\pi^4$
	$0.511866\pi^2$	$-1.2542832\pi^2$	$1.835233100\pi^4$	$0.909984\pi^2$	$-2.2298368\pi^2$	$5.800243035\pi^4$
	$-0.511866\pi^2$	$-1.2030966\pi^2$	$1.709448200\pi^4$	$-0.909984\pi^2$	$-2.140384\pi^2$	$5.409314548\pi^4$
	$-1.530477\pi^2$	$-1.0500489\pi^2$	$3.444962500\pi^4$	$-2.720848\pi^2$	$-1.86667536\pi^2$	$10.887782840\pi^4$
	$-2.5338114\pi^2$	$-0.7966677\pi^2$	$7.054879600\pi^4$	$-4.5045536\pi^2$	$-1.41629824\pi^2$	$40.701737490\pi^4$
	$-3.5118459\pi^2$	$-0.4454828\pi^2$	$12.531517000\pi^4$	$-6.24328\pi^2$	$-0.79196944\pi^2$	$39.605760750\pi^4$
	0	0	0	0	0	0

	n = 5			n = 6		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$6.8181825\pi^2$	$-3.40909\pi^2$	$58.108723140\pi^4$	$9.8181828\pi^2$	$-4.9090896\pi^2$	$120.4958742\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$9.183675\pi^2$	$-3.061225\pi^2$	$93.710985010\pi^4$	$13.224492\pi^2$	$-4.408164\pi^2$	$194.319098500\pi^4$
	0	$-3.061225\pi^2$	$9.371098501\pi^4$	0	$-4.408164\pi^2$	$19.431909850\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$10.34733\pi^2$	$-2.5868325\pi^2$	$113.758940500\pi^4$	$14.9001552\pi^2$	$-3.7250388\pi^2$	$235.890539000\pi^4$
	$3.52172\pi^2$	$-3.46725\pi^2$	$24.424334320\pi^4$	$5.0712768\pi^2$	$-4.992858\pi^2$	$50.646479390\pi^4$
	$-3.52172\pi^2$	$-2.5868325\pi^2$	$19.094214140\pi^4$	$-5.0712768\pi^2$	$-3.7250388\pi^2$	$39.593762440\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$11.04345\pi^2$	$-2.214369\pi^2$	$133.064477200\pi^4$	$15.8894568\pi^2$	$-3.1886892\pi^2$	$262.642576200\pi^4$
	$5.629005\pi^2$	$-3.33267\pi^2$	$42.792893230\pi^4$	$8.1057672\pi^2$	$-4.7990448\pi^2$	$88.734292830\pi^4$
	0	$-3.33267\pi^2$	$11.106689330\pi^4$	0	$-4.7990448\pi^2$	$23.030830990\pi^4$
	$-5.629005\pi^2$	$-2.214369\pi^2$	$36.589127360\pi^4$	$-8.1057672\pi^2$	$-3.1886892\pi^2$	$75.871209140\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$11.4865225\pi^2$	$-1.914442\pi^2$	$135.605287300\pi^4$	$16.5405924\pi^2$	$-2.7567648\pi^2$	$281.190949100\pi^4$
	$7.021465\pi^2$	$-3.08466\pi^2$	$58.815729100\pi^4$	$10.1109096\pi^2$	$-4.4419176\pi^2$	$121.961124900\pi^4$
	$2.36226\pi^2$	$-3.47838\pi^2$	$17.679399730\pi^4$	$3.4016544\pi^2$	$-5.00886\pi^2$	$36.659931160\pi^4$
	$-2.36226\pi^2$	$-3.08466\pi^2$	$15.095399620\pi^4$	$-3.4016544\pi^2$	$-4.4419176\pi^2$	$31.301884620\pi^4$
	$-7.021465\pi^2$	$-1.914442\pi^2$	$52.966058920\pi^4$	$-10.1109096\pi^2$	$-2.7567648\pi^2$	$109.830245100\pi^4$
	0	0	0	0	0	0
7	0	0	0	0	0	0
	$11.93131\pi^2$	$-1.686616\pi^2$	$145.200831800\pi^4$	$17.1810864\pi^2$	$-2.42872704\pi^2$	$301.088749000\pi^4$
	$8.006855\pi^2$	$-2.8304525\pi^2$	$72.121188350\pi^4$	$11.5298712\pi^2$	$-4.0758516\pi^2$	$149.550496200\pi^4$
	$4.044565\pi^2$	$-3.0482475\pi^2$	$25.650318860\pi^4$	$5.8241736\pi^2$	$-4.9078764\pi^2$	$58.008248880\pi^4$
	0	$-3.0482475\pi^2$	$9.291812821\pi^4$	0	$-4.9078764\pi^2$	$24.087250760\pi^4$
	$-4.044565\pi^2$	$-2.8304525\pi^2$	$24.369967390\pi^4$	$-5.8241736\pi^2$	$-4.0758516\pi^2$	$58.008248880\pi^4$
	$-8.006855\pi^2$	$-1.686616\pi^2$	$66.954400520\pi^4$	$-11.5298712\pi^2$	$-2.42872704\pi^2$	$138.836644900\pi^4$
	0	0	0	0	0	0

	n = 5			n = 6		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
8	0	0	0	0	0	0
	$12.04426\pi^2$	$-1.5055325\pi^2$	$147.330827100\pi^4$	$17.3437344\pi^2$	$-2.1679668\pi^2$	$305.505208200\pi^4$
	$8.73968\pi^2$	$-2.5979925\pi^2$	$83.131571530\pi^4$	$12.5851392\pi^2$	$-3.7411092\pi^2$	$172.381626700\pi^4$
	$5.29880\pi^2$	$-3.2603525\pi^2$	$38.707179860\pi^4$	$7.6303872\pi^2$	$-4.6949076\pi^2$	$80.264966190\pi^4$
	$1.77552\pi^2$	$-3.4822925\pi^2$	$15.278832330\pi^4$	$2.5567488\pi^2$	$-5.0145012\pi^2$	$31.682186710\pi^4$
	$-1.77552\pi^2$	$-3.2603525\pi^2$	$13.782366430\pi^4$	$-2.5567488\pi^2$	$-4.6949076\pi^2$	$28.579102680\pi^4$
	$-5.29880\pi^2$	$-2.5979925\pi^2$	$34.826864700\pi^4$	$-7.6303872\pi^2$	$-3.7411092\pi^2$	$72.218706870\pi^4$
	$-8.73968\pi^2$	$-1.5055325\pi^2$	$78.648634610\pi^4$	$-12.5851392\pi^2$	$-2.1679668\pi^2$	$163.08580870\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$12.2281425\pi^2$	$-1.3586225\pi^2$	$151.373486500\pi^4$	$17.6085252\pi^2$	$-1.95650244\pi^2$	$313.888061400\pi^4$
	$9.305415\pi^2$	$-2.392617\pi^2$	$92.315364430\pi^4$	$13.3997976\pi^2$	$-3.44536848\pi^2$	$191.425139700\pi^4$
	$6.2680525\pi^2$	$-3.0890675\pi^2$	$48.830820160\pi^4$	$9.0259956\pi^2$	$-4.4482572\pi^2$	$101.255588700\pi^4$
	$3.159425\pi^2$	$-3.43945\pi^2$	$21.811782630\pi^4$	$4.5409572\pi^2$	$-4.952808\pi^2$	$45.150599380\pi^4$
	0	$-3.43945\pi^2$	$11.829816300\pi^4$	0	$-4.952808\pi^2$	$24.530307080\pi^4$
	$-3.159425\pi^2$	$-3.0890675\pi^2$	$19.524304350\pi^4$	$-4.5409572\pi^2$	$-4.4482572\pi^2$	$40.407284410\pi^4$
	$-6.2680525\pi^2$	$-2.392617\pi^2$	$45.013098250\pi^4$	$-9.0259956\pi^2$	$-3.44536848\pi^2$	$93.396535270\pi^4$
	$-9.305415\pi^2$	$-1.35868225\pi^2$	$88.436765780\pi^4$	$-13.3997976\pi^2$	$-1.95650244\pi^2$	$183.333366270\pi^4$
	0	0	0	0	0	0
10	0	0	0	0	0	0
	$12.37495\pi^2$	$-1.23745225\pi^2$	$154.670675600\pi^4$	$17.81928\pi^2$	$-1.78193124\pi^2$	$320.700201870\pi^4$
	$9.7551275\pi^2$	$-2.212966\pi^2$	$100.059704500\pi^4$	$14.0473836\pi^2$	$-3.18667104\pi^2$	$207.483858300\pi^4$
	$7.038365\pi^2$	$-2.916800\pi^2$	$58.046304110\pi^4$	$10.1352456\pi^2$	$-4.21956\pi^2$	$120.527890000\pi^4$
	$4.251325\pi^2$	$-3.341925\pi^2$	$29.242226960\pi^4$	$6.121908\pi^2$	$-4.812372\pi^2$	$60.636681830\pi^4$
	$1.42185\pi^2$	$-3.48412\pi^2$	$14.160749600\pi^4$	$2.047464\pi^2$	$-5.0171328\pi^2$	$29.363730360\pi^4$
	$-1.42185\pi^2$	$-3.416900\pi^2$	$13.69863030\pi^4$	$-2.047464\pi^2$	$-4.815864\pi^2$	$27.384423740\pi^4$
	$-4.251325\pi^2$	$-2.916800\pi^2$	$26.581486500\pi^4$	$-6.121908\pi^2$	$-4.21956\pi^2$	$55.282444150\pi^4$
	$-7.038365\pi^2$	$-2.212966\pi^2$	$54.435773830\pi^4$	$10.135245\pi^2$	$-3.18667104\pi^2$	$112.878063500\pi^4$
	$-9.7551275\pi^2$	$-1.2374522\pi^2$	$96.693800490\pi^4$	$-14.0473836\pi^2$	$-1.7819312\pi^2$	$200.504264800\pi^4$
	0	0	0	0	0	0

	n = 7			n = 8		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$13.3636377\pi^2$	$-6.6818164\pi^2$	$223.233483000\pi^4$	$17.4545472\pi^2$	$-8.7272704\pi^2$	$380.826466600\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$18.000003\pi^2$	$-6.000001\pi^2$	$360.000048000\pi^4$	$23.510208\pi^2$	$-7.836736\pi^2$	$614.144311300\pi^4$
	0	$-6.000001\pi^2$	$36.000012000\pi^4$	0	$-7.836736\pi^2$	$614.14311300\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$20.2807668\pi^2$	$-5.0701917\pi^2$	$437.016345900\pi^4$	$26.4891648\pi^2$	$-6.6222912\pi^2$	$745.530592500\pi^4$
	$6.903512\pi^2$	$-6.7958345\pi^2$	$93.841844490\pi^4$	$9.0156032\pi^2$	$-8.876192\pi^2$	$160.067885500\pi^4$
	$-6.903512\pi^2$	$-5.0701917\pi^2$	$437.016345900\pi^4$	$-9.0156032\pi^2$	$-6.6222912\pi^2$	$125.135841800\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$21.6273162\pi^2$	$-4.34016324\pi^2$	$486.577823000\pi^4$	$28.248192\pi^2$	$-5.66878464\pi^2$	$830.095470600\pi^4$
	$11.0328498\pi^2$	$-6.5320332\pi^2$	$164.391232400\pi^4$	$14.4102528\pi^2$	$-8.5316352\pi^2$	$280.444184900\pi^4$
	0	$-6.5320332\pi^2$	$42.667457730\pi^4$	0	$-8.5316352\pi^2$	$72.788799190\pi^4$
	$-11.0328498\pi^2$	$-4.34016324\pi^2$	$140.560791700\pi^4$	$-14.4102528\pi^2$	$-5.66878464\pi^2$	$239.790071500\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$22.5135841\pi^2$	$-3.7522632\pi^2$	$520.940948100\pi^4$	$29.4054976\pi^2$	$-4.9009152\pi^2$	$888.702258900\pi^4$
	$13.762042\pi^2$	$-6.0459434\pi^2$	$225.947231600\pi^4$	$17.9749504\pi^2$	$-7.896704\pi^2$	$385.456775900\pi^4$
	$4.6300296\pi^2$	$-6.817615\pi^2$	$67.917042830\pi^4$	$6.0473856\pi^2$	$-8.90464\pi^2$	$115.863486100\pi^4$
	$-4.6300296\pi^2$	$-6.0459434\pi^2$	$57.990605690\pi^4$	$-6.0473856\pi^2$	$-7.896704\pi^2$	$98.928806660\pi^4$
	$-13.7620714\pi^2$	$-3.7522632\pi^2$	$203.474088300\pi^4$	$-17.9749504\pi^2$	$-4.9009152\pi^2$	$347.117811700\pi^4$
	0	0	0	0	0	0
7	0	0	0	0	0	0
	$23.3853676\pi^2$	$-3.30576736\pi^2$	$557.803908500\pi^4$	$30.5441536\pi^2$	$-4.3177632\pi^2$	$951.588171600\pi^4$
	$15.6934358\pi^2$	$-5.5476869\pi^2$	$277.060757100\pi^4$	$20.4975488\pi^2$	$-7.2459584\pi^2$	$472.653419900\pi^4$
	$7.9273474\pi^2$	$-6.6801651\pi^2$	$107.467442600\pi^4$	$10.3540864\pi^2$	$-8.7251136\pi^2$	$183.334712500\pi^4$
	0	$-6.6801651\pi^2$	$44.624605760\pi^4$	0	$-8.7251136\pi^2$	$76.127607330\pi^4$
	$-7.9273474\pi^2$	$-5.5476869\pi^2$	$93.619666740\pi^4$	$-10.3540864\pi^2$	$-7.2459584\pi^2$	$159.711018300\pi^4$
	$-15.6934358\pi^2$	$-3.30576736\pi^2$	$257.212025000\pi^4$	$-20.4975488\pi^2$	$4.31773696\pi^2$	$438.792359300\pi^4$
	0	0	0	0	0	0

	n = 7			n = 8		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
8	0	0	0	0	0	0
	$23.6067496\pi^2$	$-2.9508437\pi^2$	$565.986105200\pi^4$	$30.8333056\pi^2$	$-3.8541632\pi^2$	$965.547308200\pi^4$
	$17.1297728\pi^2$	$-5.54737869\pi^2$	$324.202526500\pi^4$	$22.3735808\pi^2$	$-6.6508608\pi^2$	$544.811163000\pi^4$
	$10.3858048\pi^2$	$-6.6801651\pi^2$	$152.489530500\pi^4$	$13.5651328\pi^2$	$-8.3465024\pi^2$	$253.676930200\pi^4$
	$3.4800192\pi^2$	$-6.8252833\pi^2$	$58.695162260\pi^4$	$4.5453312\pi^2$	$-8.9146688\pi^2$	$100.131355500\pi^4$
	$-3.4800192\pi^2$	$-6.68016510\pi^2$	$56.735139400\pi^4$	$-4.5453312\pi^2$	$-8.3465024\pi^2$	$90.324138030\pi^4$
	$-10.3858048\pi^2$	$-5.54737869\pi^2$	$138.638351700\pi^4$	$-13.565138\pi^2$	$-6.6508608\pi^2$	$228.246918300\pi^4$
	-17.1297728	$-2.9508437\pi^2$	$302.136594700\pi^4$	$-22.3735808\pi^2$	$-3.8541632\pi^2$	$515.431691800\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$23.9671593\pi^2$	$-2.66301721\pi^2$	$581.516385600\pi^4$	$28.73040448\pi^2$	$-3.47822656\pi^2$	$837.534201600\pi^4$
	$18.2386134\pi^2$	$-4.68952932\pi^2$	$354.638704000\pi^4$	$23.8218624\pi^2$	$-6.12509952\pi^2$	$573.606227700\pi^4$
	$12.2853829\pi^2$	$-6.04023\pi^2$	$187.415011500\pi^4$	$16.046144\pi^2$	$-7.9080128\pi^2$	$321.010240200\pi^4$
	$6.182673\pi^2$	$-6.741322\pi^2$	$83.670867730\pi^4$	$8.0728128\pi^2$	$-8.804992\pi^2$	$142.698190600\pi^4$
	0	$-6.741322\pi^2$	$45.445422310\pi^4$	0	$-8.804992\pi^2$	$77.527884120\pi^4$
	$-6.182673\pi^2$	$-6.04023\pi^2$	$74.709823880\pi^4$	$-8.0728128\pi^2$	$-7.9080128\pi^2$	$127.706972900\pi^4$
	$-12.2853829\pi^2$	$-4.68952932\pi^2$	$27.214660240\pi^4$	$-16.046144\pi^2$	$-6.12509952\pi^2$	$294.995581400\pi^4$
	$-18.2386134\pi^2$	$-2.66301721\pi^2$	$339.738679400\pi^4$	0	0	0
	0	0	0			
10	0	0	0	0	0	0
	$22.040200\pi^2$	$-2.42540641\pi^2$	$491.653012300\pi^4$	$31.67872\pi^2$	$-3.16787776\pi^2$	$1013.576750000\pi^4$
	$19.15510499\pi^2$	$-4.33741336\pi^2$	$385.731201800\pi^4$	$24.9731264\pi^2$	$-5.66519296\pi^2$	$655.751453500\pi^4$
	$19.1200499\pi^2$	$-5.7169329\pi^2$	$398.259630000\pi^4$	$18.0182144\pi^2$	$-7.4670144\pi^2$	$380.412354200\pi^4$
	$8.332597\pi^2$	$-6.5501926\pi^2$	$112.337195900\pi^4$	$10.883392\pi^2$	$-8.5553536\pi^2$	$191.642296600\pi^4$
	$2.786826\pi^2$	$-6.8288752\pi^2$	$54.399935650\pi^4$	$3.639936\pi^2$	$-8.9193472\pi^2$	$92.803888560\pi^4$
	$-2.786826\pi^2$	$-6.5501926\pi^2$	$50.671422250\pi^4$	$-3.639936\pi^2$	$-8.561536\pi^2$	$86.549032760\pi^4$
	$-8.332597\pi^2$	$-5.7169329\pi^2$	$102.115494500\pi^4$	$-10.883392\pi^2$	$-7.467008\pi^2$	$174.204429900\pi^4$
	$-19.1200499\pi^2$	$-4.33741336\pi^2$	$384.389462800\pi^4$	$-18.0182144\pi^2$	$-5.66519296\pi^2$	$356.750461400\pi^4$
	$-19.15510499\pi^2$	$-2.42540641\pi^2$	$372.645308500\pi^4$	$-24.9731264\pi^2$	$-3.16787776\pi^2$	$633.692491700\pi^4$
	0	0	0	0	0	0

	n = 9			n = 10		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$22.0909113\pi^2$	$-11.0454516\pi^2$	$610.010363100\pi^4$	$27.27273\pi^2$	$-13.6363\pi^2$	$929.750479300\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$29.755107\pi^2$	$-9.918369\pi^2$	$983.740436200\pi^4$	$36.7347\pi^2$	$-12.2449\pi^2$	$1499.375760000\pi^4$
	0	$-9.918369\pi^2$	$98.374043620\pi^4$	0	$-12.2449\pi^2$	$149.937576000\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$33.5256492\pi^2$	$-8.3813373\pi^2$	$1194.195854000\pi^4$	$41.38932\pi^2$	$-10.34733\pi^2$	$1820.143048000\pi^4$
	$11.4103728\pi^2$	$-11.2339305\pi^2$	$256.397801900\pi^4$	$14.08688\pi^2$	$-13.86905\pi^2$	$390.790736000\pi^4$
	$-11.4103728\pi^2$	$-8.3813373\pi^2$	$200.443422400\pi^4$	$-14.08688\pi^2$	$-10.34733\pi^2$	$305.507426300\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$35.7512778\pi^2$	$-7.17455556\pi^2$	$1329.628112000\pi^4$	$44.13738\pi^2$	$-8.857476\pi^2$	$2026.563194000\pi^4$
	$18.2379762\pi^2$	$-10.7978508\pi^2$	$449.217357800\pi^4$	$22.51602\pi^2$	$-13.33068\pi^2$	$684.678185900\pi^4$
	0	$-10.7978508\pi^2$	$116.593581900\pi^4$	0	$-13.33068\pi^2$	$177.707039200\pi^4$
	$-18.2379762\pi^2$	$-7.17455556\pi^2$	$384.097892000\pi^4$	$-22.51602\pi^2$	$-8.857476\pi^2$	$585.426037700\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$37.2163329\pi^2$	$-6.2027208\pi^2$	$1423.529180000\pi^4$	$45.94609\pi^2$	$-7.657682\pi^2$	$2169.683280000\pi^4$
	$22.7495466\pi^2$	$-9.9943146\pi^2$	$617.428194800\pi^4$	$28.08586\pi^2$	$-12.33866\pi^2$	$941.058062500\pi^4$
	$7.6537224\pi^2$	$-11.269935\pi^2$	$185.590901500\pi^4$	$9.44904\pi^2$	$-13.9135\pi^2$	$282.869839200\pi^4$
	$-7.653722\pi^2$	$-9.9943146\pi^2$	$158.465784800\pi^4$	$-9.44904\pi^2$	$-12.33866\pi^2$	$241.526887500\pi^4$
	$-22.7495466\pi^2$	$-6.2027208\pi^2$	$556.015615800\pi^4$	$-28.08586\pi^2$	$-7.657682\pi^2$	$847.455625600\pi^4$
	0	0	0	0	0	0
7	0	0	0	0	0	0
	$38.6575744\pi^2$	$-5.47921584\pi^2$	$969.908649800\pi^4$	$47.22524\pi^2$	$-6.746464\pi^2$	$2275.738070000\pi^4$
	$25.942210\pi^2$	$-9.1706661\pi^2$	$757.099282900\pi^4$	$32.02742\pi^2$	$-11.32181\pi^2$	$1153.939014000\pi^4$
	$13.1043906\pi^2$	$-11.0427219\pi^2$	$293.666790000\pi^4$	$16.17826\pi^2$	$-13.63299\pi^2$	$447.594513000\pi^4$
	0	$-11.0427219\pi^2$	$121.941707000\pi^4$	0	$-13.63299\pi^2$	$185.858416300\pi^4$
	$-13.1043906\pi^2$	$-9.1706661\pi^2$	$255.826169700\pi^4$	$-16.17826\pi^2$	$-11.32181\pi^2$	$389.919478300\pi^4$
	$-25.942270\pi^2$	$-5.47921584\pi^2$	$703.020065900\pi^4$	$-32.02742\pi^2$	$-6.746464\pi^2$	$1071.270408000\pi^4$
	0	0	0	0	0	0

	n = 10			n = 9		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
8	0	0	0	0	0	0
	$48.17704\pi^2$	$-6.00213\pi^2$	$2357.052748000\pi^4$	$39.0234024\pi^2$	$-4.8779253\pi^2$	$1546.620090000\pi^4$
	$34.95872\pi^2$	$-10.37197\pi^2$	$1329.689866000\pi^4$	$28.3165632\pi^2$	$-8.9417439\pi^2$	$881.782535400\pi^4$
	$21.19552\pi^2$	$-13.04141\pi^2$	$619.328442900\pi^4$	$17.168355\pi^2$	$-10.563542\pi^2$	$406.340833000\pi^4$
	$7.10208\pi^2$	$-13.92917\pi^2$	$244.461317200\pi^4$	$5.767848\pi^2$	$-11.2826277\pi^2$	$160.565758400\pi^4$
	$-7.10208\pi^2$	$-13.04141\pi^2$	$2205.517915100\pi^4$	$-5.767848\pi^2$	$-10.56342\pi^2$	$144.856490100\pi^4$
	$-21.19552\pi^2$	$-10.39197\pi^2$	$557.243105600\pi^4$	$-17.168355\pi^2$	$-8.9417439\pi^2$	$374.707197400\pi^4$
	$-34.95872\pi^2$	$-6.02213\pi^2$	$1258.378154000\pi^4$	$-28.3165632\pi^2$	$-4.8779253\pi^2$	$825.621906700\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$48.91257\pi^2$	$-5.434729\pi^2$	$2421.975783000\pi^4$	$39.6191817\pi^2$	$-4.813049\pi^2$	$1592.844999000\pi^4$
	$37.22166\pi^2$	$-9.570468\pi^2$	$1477.045831000\pi^4$	$30.1495446\pi^2$	$-7.752757\pi^2$	$969.089717200\pi^4$
	$25.09221\pi^2$	$-12.3562\pi^2$	$782.289160900\pi^4$	$20.3084901\pi^2$	$-10.0085787\pi^2$	$506.175795200\pi^4$
	$12.613777\pi^2$	$-13.7578\pi^2$	$348.384431000\pi^4$	$9.831537\pi^2$	$-10.99818\pi^2$	$17.619083100\pi^4$
	0	$-13.7578\pi^2$	$189.277060800\pi^4$	0	$-10.99818\pi^2$	$120.959963300\pi^4$
	$-12.613777\pi^2$	$-12.35627\pi^2$	$311.784778500\pi^4$	$-9.831537\pi^2$	$-10.0085787\pi^2$	$196.830767400\pi^4$
	$-25.09221\pi^2$	$-9.570468\pi^2$	$721.212860400\pi^4$	$-20.3084901\pi^2$	$-7.7520757\pi^2$	$472.529447800\pi^4$
	$-37.22166\pi^2$	$-5.434729\pi^2$	$1414.988252000\pi^4$	$-30.3084901\pi^2$	$-4.8779253\pi^2$	$942.398727400\pi^4$
	0	0	0	0	0	0
10	0	0	0	0	0	0
	$49.49809\pi^2$	$-4.949809\pi^2$	$2474.561523000\pi^4$	$40.09338\pi^2$	$-4.00934529\pi^2$	$1623.553969000\pi^4$
	$39.02051\pi^2$	$-8.851864\pi^2$	$1600.955697000\pi^4$	$31.6066131\pi^2$	$-7.17000984\pi^2$	$1050.38703300\pi^4$
	$28.1534\pi^2$	$-11.66721\pi^2$	$928.7410992000\pi^4$	$22.804254\pi^2$	$-9.4504401\pi^2$	$609.344818600\pi^4$
	$17.0053\pi^2$	$-13.36774\pi^2$	$467.876700800\pi^4$	$13.774299\pi^2$	$-10.8278694\pi^2$	$306.974068700\pi^4$
	$5.68740\pi^2$	$-13.93648\pi^2$	$226.5719936000\pi^4$	$4.60647\pi^2$	$-11.2885488\pi^2$	$148.650899900\pi^4$
	$-17.0053\pi^2$	$-13.36774\pi^2$	$467.876727500\pi^4$	$-4.60647\pi^2$	$-10.8354401\pi^2$	$138.626328000\pi^4$
	0	$-11.66721\pi^2$	$136.123789200\pi^4$	$13.774299\pi^2$	$-9.4504401\pi^2$	$279.042131000\pi^4$
	$-28.15346\pi^2$	$-8.851864\pi^2$	$870.972806200\pi^4$	$-22.804254\pi^2$	$-7.17000984\pi^2$	$571.444240600\pi^4$
	$39.02051\pi^2$	$-4.949808\pi^2$	$1547.100800000\pi^4$	$31.6066131\pi^2$	$-4.00934529\pi^2$	$1015.052841000\pi^4$
	0	0	0	0	0	0

Conclusion

The optimal control, optimal state and the optimal cost functional at varied plane profile n=1 through n=10 were obtained. The numerical solutions depict steady increase in the cost functional as the space dimension increases while the controls get smaller as the number of elements increase. The controls maintain symmetry along their minimum points at various levels of discretization. The states always shrink at the second node. This result will enhance further computational processes towards the derivation of the optimal control, state and cost functional at various spatial planes.

References

Bawa M 2013. Implementation of the finite element technique to the optimal control of one dimensional energized wave equation. *Int. Org. Scientific Res. (IOSR) J. Mathematics,* 5(6): 31-38.

Binder L 1911. Uber Aussere Wurmleitung und Erwrumung Elektrisc her Maschinen, *Dissertation*, *Technische Hoschudle*. W. Knapp Verlag, Halle, Munchen, Germany.

Duchateau P & Zachmann DW 1986. *Partial Differential Equations*, 2nded. McGraw Hill, New York, NY, USA.

Pain HJ 1976.*The physics of vibrations and waves,* 2nd edition, John Wiley and Sons.

Raisinghania MD 2010. *Advanced Differential Equations.* Chand and Company Ltd, New Delhi.

Rao SS 1989. *The Finite Element Method in Engineering.* Pergamon Press.

Reju SA 1995. *Computational Optimization in Mathematical Physics;* Ph.D. Thesis, Univ. of Ilorin, Ilorin, Nigeria.

Schmidt EA 1924. *Fuppl.Festschrift*. Springer Verlag, Berlin, Germany.

Singh MA & Titli AJ 1978. *System Decomposition*, *Optimization & Control*. Pergamon, Toulouse, France.